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Some Inequalities for Derivatives of Polynomials

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If $p(z)$ is a polynomial of degree at most n having no zeros in $|z|<1$, then according to a well known result conjectured by Erdős and proved by Lax $\max_{|z|=1}|p'(z)| \leqslant (n/2)\max_{|z|=1}|p(z)|$. On the other hand, by a result due to Turan, if $p(z)$ has all its zeros in $|z|\leqslant 1$, then $\max_{|z|=1}|p'(z)| \geqslant (n/2)\max_{|z|=1}|p(z)|$. In this paper we generalize and sharpen these inequalities.

1. INTRODUCTION AND STATEMENT OF RESULTS

If $p(z)$ is a polynomial of degree at most n, then according to a famous result known as Bernstein's inequality (for references see [6])

$$\max_{|z|=1} |p'(z)| \leqslant n \max_{|z|=1} |p(z)|. \tag{1.1}$$

Here equality holds if and only if $p(z)$ has all its zeros at the origin. In case $p(z)$ does not vanish in $|z|<1$, it was conjectured by Erdős and proved by Lax [4] that (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

On the other hand it was proved by Turan [7] that if $p(z)$ has all its zeros in $|z|\leqslant 1$, then

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.3}$$

Both the above inequalities are sharp and become equalities for $p(z)=\lambda+\mu z^n$, $|\lambda|=|\mu|$.

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Recently Azis and Dawood [1] improved inequalities (1.2) and (1.3) by proving

THEOREM A [1, Theorem 2]. *If $p(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2} \{\max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)|\}. \tag{1.4}$$

THEOREM B [1, Theorem 4]. *If $p(z)$ is a polynomial of degree n which has all its zeros in $|z| \leqslant 1$, then*

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{2} \{\max_{|z|=1} |p(z)| + \min_{|z|=1} |p(z)|\}. \tag{1.5}$$

Here we generalize the above theorems by proving the following more general.

THEOREM 1. *If $p(z)$ is a polynomial of degree n having no zeros in $|z|<K$, $K \geqslant 1$, then*

$$\max_{|z|=1} |p^{(s)}(z)| \leqslant \frac{n(n-1)\cdots(n-s+1)}{1+K^s} (\max_{|z|=1} |p(z)| - \min_{|z|=K} |p(z)|). \tag{1.6}$$

THEOREM 2. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leqslant K$, then*

$$\max_{|z|=1} |p'(z)| \geqslant \left(\frac{n}{1+K}\right) \max_{|z|=1} |p(z)| + \frac{n}{K^{n-1}(1+K)} \min_{|z|=K} |p(z)| \tag{1.7}$$

if $K \leqslant 1$, and

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{(1+K^n)} (\max_{|z|=1} |p(z)| + \min_{|z|=K} |p(z)|) \tag{1.8}$$

if $K \geqslant 1$.

Both these inequalities are best possible. In (1.7) equality holds for $p(z)=(z+K)^n$ and in (1.8) for $p(z)=z^n+K^n$.

As is immediate to see, Theorem 1 sharpens a result of Govil and Rahman [3, Theorem 4]. If we take $s=1$ in Theorem 1, we get the following result which sharpens a result of Malik [5].

COROLLARY 1. *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < K$, $K \geqslant 1$, then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+K} (\max_{|z|=1} |p(z)| - \min_{|z|=K} |p(z)|). \tag{1.9}$$

The result is best possible and the equality holds for $p(z) = (z+K)^n$.

Theorem A of Aziz and Dawood [1] is a special case of the above Corollary when $K=1$. If we take $K=1$ in Theorem 2, we get Theorem B of Aziz and Dawood [1]. In general Theorem 2 sharpens results of Govil [2] and Malik [5].

Remark. In all the above inequalities (1.6), (1.7), (1.8), and (1.9), it is not possible to replace the expression $\min_{|z|=K} |p(z)|$ by $\min_{|z|=1} |p(z)|$, as the polynomial $p(z) = (z+K)^n$ shows for inequalities (1.6), (1.7), and (1.9) and $p(z) = z^n + K^n$ shows for the inequality (1.8).

2. LEMMAS

We need the following lemmas.

LEMMA 1. *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < K$, $K \geqslant 1$, then*

$$K^s |p^{(s)}(e^{i\theta})| \leqslant |q^{(s)}(e^{i\theta})|, \qquad 0 \leqslant \theta < 2\pi. \tag{2.1}$$

Here and elsewhere $q(z)$ stands for $z^n \{\overline{p(1/\bar{z})}\}$.

This lemma is in fact implicit in the proof of Theorem 4 of Govil and Rahman [3]; however, for the sake of completeness we give here a brief outline of the proof. For this, first let us suppose that all the zeros of $p(z)$ lie on $|z| = K \geqslant 1$. Then all the zeros of $P_1(z) = p(Kz)$ lie on $|z| = 1$ and so do the zeros of $Q_1(z) = z^n \{\overline{P_1(1/\bar{z})}\} = K^n q(z/K)$. For every λ with $|\lambda| > 1$, the polynomial $P_1(z) - \lambda Q_1(z)$ has all its zeros on $|z| = 1$; hence by the Gauss–Lucas Theorem all the zeros of the sth derivative $P_1^{(s)}(z) - \lambda Q_1^{(s)}(z)$ lie in $|z| \leqslant 1$. This implies that

$$K^s |p^{(s)}(Kz)| = |P_1^{(s)}(z)| \leqslant |Q_1^{(s)}(z)| = K^{n-s} |q^{(s)}(z/K)|$$

for $|z| \geqslant 1$. In particular we have

$$|p^{(s)}(K^2 e^{i\theta})| \leqslant K^{n-2s} |q^{(s)}(e^{i\theta})|, \qquad 0 \leqslant \theta < 2\pi. \tag{2.2}$$

The polynomial $p^{(s)}(Kz)$ is a polynomial of degree $n-s$ having all its zeros in $|z| \leqslant 1$; hence on considering the quotient

$$z^{n-s}\{\overline{p^{(s)}(K/\bar{z})}\}/p^{(s)}(Kz)$$

in $|z| \geqslant 1$ one gets easily as a consequence of the maximum modulus principle that

$$|z^{n-s}\{\overline{p^{(s)}(K/\bar{z})}\}| \leqslant |p^{(s)}(Kz)| \qquad \text{for} \quad |z| \geqslant 1,$$

which gives

$$K^{n-s}|p^{(s)}(e^{i\theta})| \leqslant |p^{(s)}(K^2 e^{i\theta})|, \qquad 0 \leqslant \theta < 2\pi. \tag{2.3}$$

Combining this with (2.2) we get (2.1) for polynomials having all their zeros on $|z| = K \geqslant 1$.

If the zeros of $p(z)$ lie in $|z| \geqslant K \geqslant 1$ but not necessarily on $|z| = K$, then for every real γ, the polynomial $p(z) + e^{i\gamma}Q_1(z/K)$ has all its zeros on $|z| = K \geqslant 1$ and applying (2.1), which has been proved for polynomials having all the zeros on $|z| = K \geqslant 1$, to the polynomial $p(z) + e^{i\gamma}Q_1(z/K)$, Lemma 1 will follow.

LEMMA 2. *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < K$, $K \geqslant 1$, and $q(z) = z^n\{\overline{p(1/\bar{z})}\}$, then for $|z| \geqslant 1/K$,*

$$|q^{(s)}(z)| \geqslant mn(n-1)\cdots(n-s+1)|z|^{n-s}, \tag{2.4}$$

where $m = \min_{|z|=K}|p(z)|$.

Proof of Lemma 2. Because the polynomial $p(z)$ has no zeros in $|z| < K$, $K \geqslant 1$, the polynomial $q(z) = z^n\{\overline{p(1/\bar{z})}\}$ has all its zeros in $|z| \leqslant 1/K \leqslant 1$. Therefore for every α, $|\alpha| < 1$, the polynomial $q(z) - \alpha m z^n$ has all its zeros in $|z| \leqslant 1/K$, which implies by the Gauss–Lucas theorem that $q^{(s)}(z) - \alpha mn(n-1)\cdots(n-s+1)\,z^{n-s}$ has all its zeros in $|z| \leqslant 1/K$ and from which (2.4) will follow.

LEMMA 3. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant K$, $K \geqslant 1$, then*

$$\max_{|z|=1}|p'(z)| \geqslant \frac{n}{1+K^n}\max_{|z|=1}|p(z)|.$$

The result is best possible with equality for $p(z) = z^n + K^n$.

The above result is due to Govil [2].

3. Proofs of Theorems

Proof of Theorem 1. Let $p(z)$ be a polynomial of degree n having all its zeros in $|z|\leqslant 1$. Then $q(z)=z^n\{\overline{p(1/\bar{z})}\}$ has no zeros in $|z|<1$; hence by Lemma 1.

$$|q^{(s)}(e^{i\theta})|\leqslant|p^{(s)}(e^{i\theta})|,\qquad 0\leqslant\theta<2\pi. \tag{3.1}$$

If $p(z)$ is a polynomial of degree n, $\|p\|=\max_{|z|=1}|p(z)|$, then by Rouché's theorem for every λ with $|\lambda|>1$, the polynomial $p(z)-\lambda\|p\|z^n$ has all its zeros in $|z|<1$; hence applying (3.1) to the polynomial $p(z)-\lambda\|p\|z^n$ we conclude that if $q(z)=z^n\{\overline{p(1/\bar{z})}\}$, then

$$|p^{(s)}(e^{i\theta})|+|q^{(s)}(e^{i\theta})|\leqslant\|p\|\,n(n-1)\cdots(n-s+1). \tag{3.2}$$

If $p(z)$ is a polynomial of degree n having no zeros in $|z|<K$, $K\geqslant 1$, and if $m=\min_{|z|=K}|p(z)|$ then for every α with $|\alpha|<1$ the polynomial $p(z)-\alpha m$ has no zeros in $|z|<K$, $K\geqslant 1$. This result is clear if $p(z)$ has a zero on $|z|=K$ for then $m=0$ and hence $p(z)-\alpha m=p(z)$. In case $p(z)$ has no zeros on $|z|=K$, then, for every α with $|\alpha|<1$, we have $|p(z)|>|\alpha|m$ on $|z|=K$ and the result follows from Rouché's theorem. Thus in any case $p(z)-\alpha m$ has no zeros in $|z|<K$, $K\geqslant 1$, and therefore applying Lemma 1 to the polynomial $p(z)-\alpha m$, we get

$$K^s|p^{(s)}(e^{i\theta})|\leqslant|q^{(s)}(e^{i\theta})-\bar{\alpha}n(n-1)\cdots(n-s+1)\,me^{i(n-s)\theta}|. \tag{3.3}$$

Choosing argument of α suitably, making $|\alpha|\to 1$, and noting that by Lemma 2, $|q^{(s)}(e^{i\theta})|\geqslant mn(n-1)\cdots(n-s+1)$, we get from (3.3)

$$K^s|p^{(s)}(e^{i\theta})|\leqslant|q^{(s)}(e^{i\theta})|-mn(n-1)\cdots(n-s+1),$$

which is clearly equivalent to

$$|q^{(s)}(e^{i\theta})|\geqslant K^s|p^{(s)}(e^{i\theta})|+mn(n-1)\cdots(n-s+1). \tag{3.4}$$

Now combining (3.4) with (3.2), Theorem 1 follows.

Proof of Theorem 2. First we prove (1.7). Since the polynomial $p(z)$ has all its zeros in $|z|\leqslant K\leqslant 1$, the polynomial $q(z)=z^n\{\overline{p(1/\bar{z})}\}$ has no zeros in $|z|<1/K$, $1/K\geqslant 1$; hence applying Theorem 1, with $s=1$, to $q(z)$ we get

$$|q'(z)|=\frac{n}{(1+1/k)}\left(\max_{|z|=1}|q(z)|-\min_{|z|=1/K}|q(z)|\right),$$

which gives that on $|z|=1$,

$$|np(z)-zp'(z)| \leqslant \frac{nK}{1+K} \max_{|z|=1} |p(z)| - \frac{nK}{1+K} \min_{|z|=1/K} |q(z)|$$
$$= \frac{nK}{1+K} \max_{|z|=1} |p(z)| - \frac{nK}{(1+K)\, K^n} \min_{|z|=K} |p(z)|,$$

which implies that for $|z|=1$,

$$n\,|p(z)| - |p'(z)| \leqslant \frac{nK}{1+K} \max_{|z|=1} |p(z)| - \frac{nK}{(1+K)\, K^n} \min_{|z|=K} |p(z)|. \quad (3.5)$$

Now choosing z_0 such that $|p(z_0)| = \max_{|z|=1} |p(z)|$, we get from (3.5)

$$|p'(z_0)| \geqslant \left(\frac{n}{1+K}\right) \max_{|z|=1} |p(z)| + \frac{nK}{(1+K)\, K^n} \min_{|z|=K} |p(z)|,$$

from which (1.7) follows.

To prove (1.8), note that if $m = \min_{|z|=K} |p(z)|$, then for every α with $|\alpha|<1$, the polynomial $p(z)+\alpha m$ has all its zeros in $|z| \leqslant K$, $K \geqslant 1$. This is clear if $p(z)$ has a zero on $|z|=K$, because in that case $m=0$ and therefore $p(z)+\alpha m = p(z)$. In case $p(z)$ has not zero on $|z|=K$, then, for every α with $|\alpha|<1$, we have $|p(z)| > m|\alpha|$ on $|z|=K$ and on applying Rouché's theorem the result will follow. Thus $p(z)+\alpha m$ has all its zeros in $|z| \leqslant K$, $K \geqslant 1$ and hence, applying Lemma 3 to $p(z)+\alpha m$, we get

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{1+K^n} \max_{|z|=1} |p(z)+\alpha m|. \quad (3.6)$$

If we choose z_0 such that $|p(z_0)| = \max_{|z|=1} |p(z)|$, (3.6) in particular gives

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{1+K^n} (|p(z_0)+\alpha m|). \quad (3.7)$$

Now choosing α so that the right hand side of (3.5) is

$$\frac{n}{1+K^n} (|p(z_0)| + |\alpha| m)$$

and making $|\alpha| \to 1$, we get (1.8).

The proof of Theorem 2 is thus complete.

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